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# MATHEMATICS

( Paper—III )

*Full Marks* : 100

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

Answer any *ten* questions of the following : 10×10=100

**1.** Suppose $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences of real numbers such that $x_n \le y_n \le z_n$, for all $n$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = c$, $c$ is a real number. Prove that $\lim_{n\to\infty} y_n = c$. Hence prove that if $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} |x_n| = 0$. Give an example to show that when $\alpha \ne 0$, $\lim_{n\to\infty} |x_n| = \alpha$ does not imply $\{x_n\}$ is convergent. 6+2+2=10

**2.** *(a)* Evaluate $\iint dx\,dy$ over the domain bounded by $y = x^2$ and $y^2 = x$.

(b) Evaluate the triple integral $\iiint\limits_R dx\,dy\,dz$ where

$$R = \{(x, y, z) : x^2 + y^2 + z^2 \le r^2, r > 0\}$$

5+5=10

3. (a) Prove that the product of the length of the perpendiculars from $(x_1, y_1)$ to the lines represented by $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$$

(b) Show that

$$x^2 + 6xy + 9y^2 - 5x - 15y + 6 = 0$$

represents a pair of parallel straight lines.

5+5=10

4. (a) Determine the number of elements of order 5 in $Z_{25} \oplus Z_5$ by listing the elements with justification.

(b) Prove that the centre of a group is a normal subgroup of the group. 4+6=10

5. (a) Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

(b) Find the range and Kernel of $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x + z,\ x + y + 2z,\ 2x + y + 3z)$$

5+5=10

6. (a) If $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$, then show that infimum of $S$ is 0.

(b) The sequence $(1 + (-1)^n)$ is not a Cauchy sequence. Justify.

(c) Prove that $\lim_{x \to 0} \left\{ \cos\left(\frac{1}{x}\right) \right\}$ does not exist

but that $\lim_{x \to 0} \left\{ x \cos\left(\frac{1}{x}\right) \right\} = 0$. 3+2+5=10

7. (a) If $A$, $B$, $A + B$ are each non-singular matrices, then prove that

$$A(A+B)^{-1}B = B(A+B)^{-1}A = (A^{-1} + B^{-1})^{-1}$$

(b) Let $\mathbb{R}^3$ be the vector space over $\mathbb{R}$, with respect to usual vector addition and scalar multiplication. Determine whether or not the set

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \right\}$$

is linearly independent. 6+4=10

**8.** (a) Prove that if $\gcd(a, b) = d$, then the equation $ax + by = c$ has an integer solution if and only if $d \mid c$.

(b) Find all integer solutions of the following equation, or show that none exist :

$$21x + 13y = 1791$$ 6+4=10

**9.** (a) Solve the following linear initial-value problem :

$$\frac{dy}{dx} - 3y = -9x, \; y(2) = 13$$

(b) In a certain culture of bacteria, the rate of increase is proportional to the number present. If it is found that their number doubles in 4 hours, what should be their number at the end of 12 hours?

(c) Find the Wronskian of the set $\{1-x, 1+x, 1-3x\}$. 4+4+2=10

10. (a) Solve :

$$(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)dz = 0$$

(b) Eliminate arbitrary constants $a$ and $b$ from $z = (x-a)^2 + (y-b)^2$ to form a partial differential equation.

(c) Solve :

$$z\frac{\partial z}{\partial x} + x = 0$$

4+3+3=10

11. (a) Find the directional derivative of the function $f(x, y, z) = x^2 - y^2 + z^2$ at $P(1, 2, -3)$ in the direction of $\overrightarrow{PQ}$, where the coordinates of $Q$ are $(3, 1, 2)$.

(b) Determine $a$, $b$, $c$ so that the vector

$$\vec{u} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy - 2z)\hat{k}$$

is irrotational.

(c) Given that $f(0) = 1$, $f(1) = 3$, $f(3) = 55$. Find a quadratic polynomial that fits the given data. 3+3+4=10

12. (a) Show that the function $f(z) = \sqrt{|xy|}$ is not differentiable at the origin, although Cauchy-Riemann equations are satisfied at that point.

(b) Evaluate

$$\oint_C \frac{e^{2z}}{(z+1)^4} dz$$

where $C$ is the circle $|z| = 3$. 6+4=10

13. (a) Prove that the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

(b) If a plane passes through a fixed point $(\alpha, \beta, \gamma)$ and cuts the axes at $P$, $Q$, $R$; show that the locus of the centre of the sphere passing through the origin and the points $P$, $Q$, $R$ is

$$\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = 2$$

5+5=10

14. (a) Find the solution of

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 3e^{-x} - 10\cos(3x)$$

*(b)* Evaluate :

$$\lim_{z \to i} \frac{3z^4 - 3z^3 + 8z^2 - 2z + 5}{z - i}$$

*(c)* Solve :

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x + y$$

4+3+3=10

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